Type of self-organized criticality model based on neural networks

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Based on the standard self-organizing map neural network model, we introduce a kind of coupled map lattice system to investigate self-organized criticality (SOC) in the activity of model neural populations. Our system is simulated by a more detailed integrate-and-fire mechanism and a kind of local perturbation driving rule; it can display SOC behavior in a certain range of system parameters, even with period boundary condition. More importantly, when the influence of synaptic plasticity is adequately considered, we can find that our system's learning process plays a promotive role in the emergence of SOC behavior.

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I. INTRODUCTION

A few years ago, Bak *et al.* introduced the concept of the "self-organized criticality" (SOC) in the sand pile model [1]. From then on, this concept has been widely studied in many extended dissipative dynamical systems, such as earthquakes [2], biology evolution [3], forest fires [4], and so on. It is shown that all these large dynamical systems tend to self-organize into a statistically stationary state without intrinsic spatial and temporal scales. This critical state is achieved over a wide range of the parameters of the system and no fine tuning is needed, and it is characterized by a power-law distribution of avalanche sizes, where the size is the total number of toppling events or unstable units. In this sense, the dynamical state of every spatial and temporal scale can emerge if the system is in a SOC state. Therefore, it is considered that the system in a SOC state has maximal complexity and latent computing potency.

The human brain is one of the most complex systems, and it possesses about $10^{10}-10^{12}$ neurons. The brain's information process has the properties of stability and variability—on the one hand, there is relative stable information stored in the brain and on the other hand, the brain is influenced by the environment and people should continuous-update knowledge and points. So it is stated that the brain must operate at the critical state. This view is consistent with Bak's view [5], he argues that the brain can be neither subcritical nor supercritical but self-organized critical; in the first case, the external input signal can only access a small limited part of the information; in the second case, any input would cause an explosive branching process within the brain, and connect the input with essentially everything that is stored in the brain; except in last case, the brain has an appropriate sensitivity to small shocks. In fact, the strong analogies between the dynamics of the SOC model for earthquakes and that of neurobiology have been realized by Hopfield [6], and some work (including ourselves) has been done [7–9]. We hope that grasping the mechanics of SOC processes in the brain will be helpful in understanding the higher functions of the brain. So we try to study SOC in the activity of model neural populations.

In this paper, we introduce a kind of coupled map lattice system to simulate the associative memory process of brain. Our system has a kind of local perturbation driving rule and a more detailed integrate-and-fire mechanism that are different from previous systems. We think that the local driving is more fit for the simulation of the associative memory process of brain because it is natural to assume that not all the neurons in cortex but some special neurons in a particular location of the cortex respond the external signal. Our system can emerge SOC behavior (which is characterized by the power-law behavior of the avalanche size) in a certain range of system parameters, even with a period boundary condition. More importantly, when the influence of synaptic plasticity is adequately considered, we can find that our system's learning process plays a promotive role in the emergence of SOC behavior. We also find the average activity level of synapses have important influence on the SOC behavior.

II. THE MODEL

Our model is a kind of coupled map lattice system based on the standard self-organizing map (SOM) model [10]. It has two layers, the first one is an input layer, there are h neurons, receiving an h dimension input vector $\vec{\zeta}$. The second one is a computing layer, it is a two-dimensional square lattice with $L\times L$ neurons, where every neuron is connected with h input neurons, and the afferent weight vector $\vec{\omega}_{ij}$ is an h-dimensional vector.

During a learning step t, an h-dimensional vector $\vec{\zeta}$ is input, and the computing layer finds the "winner" neuron (i^*,j^*) which has the best-matching weight vector, according to the formula:

$$\|\vec{\zeta}(t)-\vec{\omega}_{i^*j^*}(t)\|=\min_{i,j}\|\vec{\zeta}(t)-\vec{\omega}_{ij}(t)\|, \tag{1}$$

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where the term $\|\vec{\zeta}(t)-\vec{\omega}_{ij}(t)\|$ is the distance between $\vec{\zeta}(t)$ and $\vec{\omega}_{ij}(t)$. All neuron outputs are 0 except for the winner's output η_{i*j*}, which is 1. Then the weight vectors of all the neurons within the certain neighborhood $N_{i*j*}(t)$ of the winning neuron $(i*,j*)$ are updated using the Kohonen rule. Other's are not updated. The updating rule is written as follows:

$$\vec{\omega}_{ij}(t+1)=\begin{cases}\vec{\omega}_{ij}(t)+1\times[\vec{\zeta}(t)-\vec{\omega}_{i*j*}(t)], & \text{if } (i,j)=(i*,j*),\\ \vec{\omega}_{ij}(t)+0.5\times[\vec{\zeta}(t)-\vec{\omega}_{i*j*}(t)], & \text{if } (i,j)\in N_{i*j*}(t)\\ & \quad\text{and } (i,j)\neq(i*,j*),\\ \vec{\omega}_{ij}(t), & \text{otherwise,}\end{cases} \tag{2}$$

where $N_{i*j*}(t)$ is a time variable, it is very wide in the beginning and shrinks monotonously with time until it only includes a winner.

The aforementioned is our system's learning mechanism, and after this process, we can find that when a specific vector $\vec{\zeta}$ is input, only one particular neuron responds, different inputs have different response neurons (winners), similar inputs have near winners, and the input weights self-organize into a topological map of the input space [see Fig. 2(b)]. That means the output state of the neural network is evolved from the disordered case to the stable and topology preserving state in state space.

According to the neuron-dynamical picture of the brain, the essential feature of the associative memory process can be described as a kind of integrate-and-fire process [7].

When a specific external input signal is input into brain, a particular location of cortex will respond (the membrane potential of response neurons increase). When the membrane potential of a neuron exceeds the threshold, the neuron sends out signals in the form of action potentials and then returns to the rest state (the neuron fires). The signal is transferred by the synapses to the other neurons, which has an excitatory or inhibitory influence on the membrane potential of the receiving cells according to whether the synapses are excitatory or inhibitory, respectively. The resulting potential, if it also exceeds the threshold, leads to the next step firing, and so on, giving an avalanche. It will then cause a response in some other areas of the cortex.

To grasp the associative memory mechanism and to do the simulation in our model, we add a kind of integrate-and-fire mechanism into our model, described in the following detail.

In this mechanism, we only consider the computing layer (the square lattice), representing a sheet of cells occurring in the cortex. For any neuron sited at position (i,j) in the lattice, we give it a dynamical variable V_{ij}, which represents the membrane potential. $V_{ij}=0$ and $V_{ij}>0$ represent the neuron in a rest state and depolarized state respectively. Here we do not consider the situation of $V_{ij}<0$, which represents the neuron in the hyperpolarized state.

When a neuron's dynamical variable V_{i*j*} exceeds a threshold $V_{\mathrm{th}}=1$, the neuron $(i*,j*)$ is unstable and it will fire and return to a rest state (V_{i*j*} returns to zero). Each of the nearest four neighbors will receive a pulse (action potential) and its membrane potential $V_{i'j'}$ will be changed

$$V_{i'j'}\rightarrow V_{i'j'}+\alpha*\beta*V_{i*j*}$$

$$V_{i*j*}\rightarrow 0, \tag{3}$$

where the term $\alpha\times\beta\times V_{i*j*}$ represents the action potential between the neuron that fires and its neighbors. We assume it is proportional to V_{i*j*}. Here, we assume that the parameter α represents the average activity level of synapse between neurons in someone's brain, and it can be tuned to simulate the difference between individuals (e.g., clever or foolish). We assume β represents the influence of synaptic plasticity. It can be described by a general Hebbian rule: If two neighbor neuron respond states for a specific input pattern are similar, the synapse connection between them is strong, otherwise it is weak. So we let β be a function, $\beta=\exp(-\|\vec{\omega}_{i*j*}-\vec{\omega}_{i'j'}\|^2)$, where the term $\|\vec{\omega}_{i*j*}-\vec{\omega}_{i'j'}\|$ is the distance between the firing neuron $(i*,j*)$ and neuron (i',j') in the input weight space. β can also have another function form, if the form can also represent the general Hebbian rule.

In study of SOC the effect of boundary conditions have been widely discussed [11–14]. In our model, we generally use open boundary condition, but for further discussion, we also use the period boundary condition in Sec. III D.

Now, we present the computer-simulation procedure of this model in detail.

(i) Variable initialization—Here, we let $h=2$. In the two-dimensional input space, randomly create many input vectors, both components of the input vector are between [0,1]. Randomly initialize the afferent weight vectors between [0, 1; 0,1] too. Let the dynamical variables V_{ij} randomly distribute between [0,1].

(ii) Learning process—During each learning step, a single vector is chosen randomly from the input vectors, and into the network, then the winner neuron is found according to formula (1), and the afferent weight vectors are updated according to formula (2). After some steps (the number of the learning steps increases with the increment of the lattice size L), the state of the network reaches a stable and topology preserving case, and the topological structure of the input's space has been learned and stored in the model.

(iii) Associative memory and avalanche process.—Here we use the sequential update mechanism.

(a) Driving Rule—Randomly choose a single input vector, find the winner neuron (i^*, j^*), and let its dynamical variable $V_{i^*j^*}=1.0$. The winner is unstable and will fire, and then an avalanche (associate memory) begins.

(b) If there exits any unstable neuron, $V_{ij} \geqslant V_{\rm th}=1$, redistribute the dynamical variables of it and its nearest neighbors according to Eq. (3).

(c) Repeat step (b) until all the neurons of the lattice are stable. Define this process as one avalanche, and define the avalanche size (associate memory size) as the number of all unstable neurons in this process.

(d) Begin step (a) again and another new avalanche (associate memory) begins.

It is worth noting that our integrate-and-fire mechanism [expression (3)] is similar to the dynamical rule of the OFC earthquake model when we treat $\alpha \times \beta$ as one parameter [2]. But there are important differences between them. Our mechanism has more details and we think that these details are important to simulate associative memory processes. They are the natural choice for our model.

Our driving rule [step (a)] is a kind of local perturbation rule. It is different from the global perturbation rule in previous systems [7–9] and the OFC model [2]. We think, it is impossible that a specific external signal will cause the incremention of all the neuron's membrane potential of the cortex at the same time. The situation should be that the neurons in a particular location of cortex respond to the external signal, then bring change to neuron states of another area (just like we mentioned above). So we use the local driving rule. We think it is more fit for the simulation of associative memory processes of the brain.

III. SIMULATION RESULTS

A. Power-law behavior and influence of the learning process

First, we let the size of the lattice be 20×20, where $\alpha = 0.26$ is fixed. Here we consider two situations: one is without learning process, the other is learning some steps. In these two situations, the probability of the avalanche size (unstable neurons) $P(S)$ as a function of size S can be seen in Fig. 1. We find that in the associated memory process, for a learning situation, the distribution of the avalanche size satisfies the power-law behavior $P(S)\propto S^{-\tau}$, $\tau\approx 1.12$. We can also see, without the learning situation, there is no power-law behavior but a localized behavior.

The trademark of SOC is the existence of a power-law distribution function of the avalanche sizes that scale with the system size [2]. With the learning process our system can emerge as power-law behavior, and as we can see in detail in Sec. III C the large size cutoff of the avalanche distribution scales with the system size L. So we conclude that our system can display SOC behavior with the learning process.

We think that the reason for the difference between the two conditions is the influence of synaptic plasticity (the parameter β). Before the learning process, the afferent

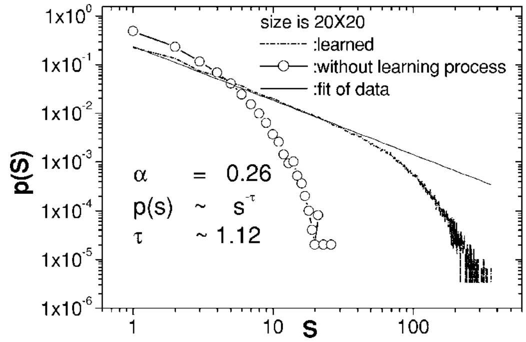

FIG. 1. The probability of the avalanche size (unstable neurons) $P(S)$ as a function of size S for a 20×20 system with $\alpha=0.26$. The curves correspond to two conditions: one is with learning some steps, another is without the learning process.

weights are randomly distributed; they do not respond to the topological structure of the input space [see Fig. 2(a)]. So $\beta=\exp(-\|\vec{\omega}_{i^*j^*}-\vec{\omega}_{i'j'}\|^2)$ is small, which means the synaptic connections between neighbor neurons is weak. After the learning process, the neighbor neurons responded to the near input vectors, and the afferent weights self-organized into a topological map of the input space see Fig. 2(b). So $\beta = \exp(-\|\vec{\omega}_{i^*j^*}-\vec{\omega}_{i'j'}\|^2)$ is large, which means the synaptic connections between the neighbor neurons are strong. As we can see in next subsection, as the synaptic connection strength increases, the SOC behavior becomes more obvious.

So the learning process can strengthen the synaptic connection and play a promotive role in the emergence of SOC behavior. We can think that, before learning, the system operates at a subcritical state, in which case the external input signal can only access a small, limited part of the system. After the learning process, the system is trained to operate at the SOC state, which is highly sensitive to small outer input and robust. It is consistent with PBak's view—the brain can neither be subcritical nor supercritical, it must be critical and self-organized [5]. We think it is consistent with the real world too, since if we do not learn, how can we associate one thing with another thing? So for further investigation, we use the condition with the learning process below.

B. Influence of the pulse discharging intensity parameter α

In our simulations, we assume α represents the average activity level of strength of synapse and it can be tuned to simulate the difference between individuals. Because there are intelligent differences exactly between individuals, our consideration of α in our model is natural. As shown in Figs. 3 and 4, when α is small ($\alpha<0.24$), the probability of the avalanche size $P(S)$ decays exponentially with the size of the avalanches and the biggest avalanche sizes do not scale with the system size L, which means there are only localized behaviors. With α increasing, the power-law behavior gradually generates, and the transition from localized to SOC behavior occurs (near the point of $\alpha=0.24$), but when α is very large ($\alpha>0.26$), the associative memory process falls into a dead loop state. (That means the avalanche process will not stop, and some neurons will fire again and again, just like the term "dead loop" in computer programming.) As we can see

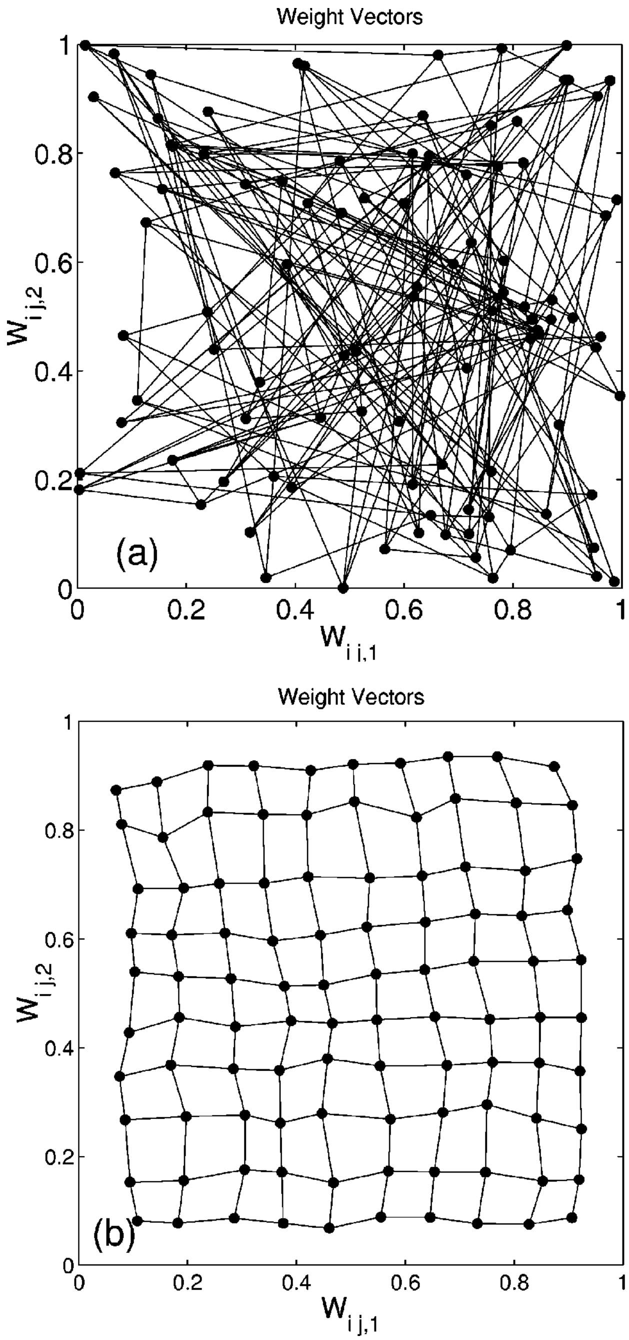

FIG. 2. Self-organization of the afferent weights. The weight vector of each neuron in the 10×10 lattice is plotted as a point in the original input space; each weight vector is connected to those of the four neighbors by a line. (a) Before the learning process and (b) after the learning process.

in detail in Figs. 4, 5, and Sec. III C, the large size cutoff of the avalanche distribution scales with the system size L in the range of $0.24<\alpha<0.26$, and we can conclude that the system can display SOC behavior in this particular range of parameter α.

C. Influence of lattice size

In the OFC model, the lattice size L is important. If the large size cutoff of the avalanche distribution does not scale

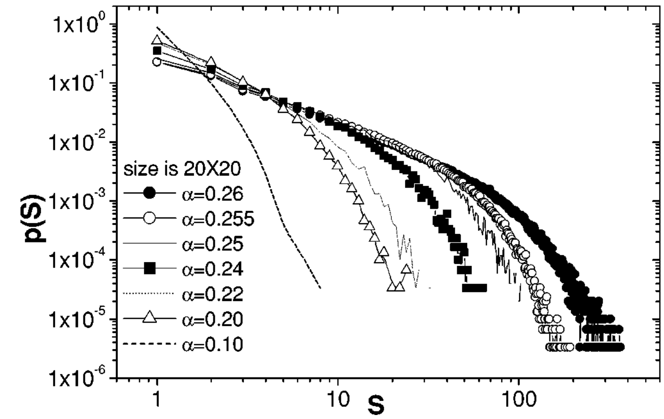

FIG. 3. Same as Fig. 1 with the learning process condition, but with parameter α changed.

with the system size L, it is a localized behavior, otherwise it means SOC. We have investigated the influence of lattice size. In Fig. 4(a), we show the results of simulations with $L=15$, 25, and 40, respectively, when $\alpha=0.20$. We can see that all the biggest avalanche sizes are about 20–30 and they do not scale with the system L; it is a localized behavior.

In Fig. 4(b), we show the results of simulations with $L=15$, 25, and 40, respectively, when $\alpha=0.255$. We can see

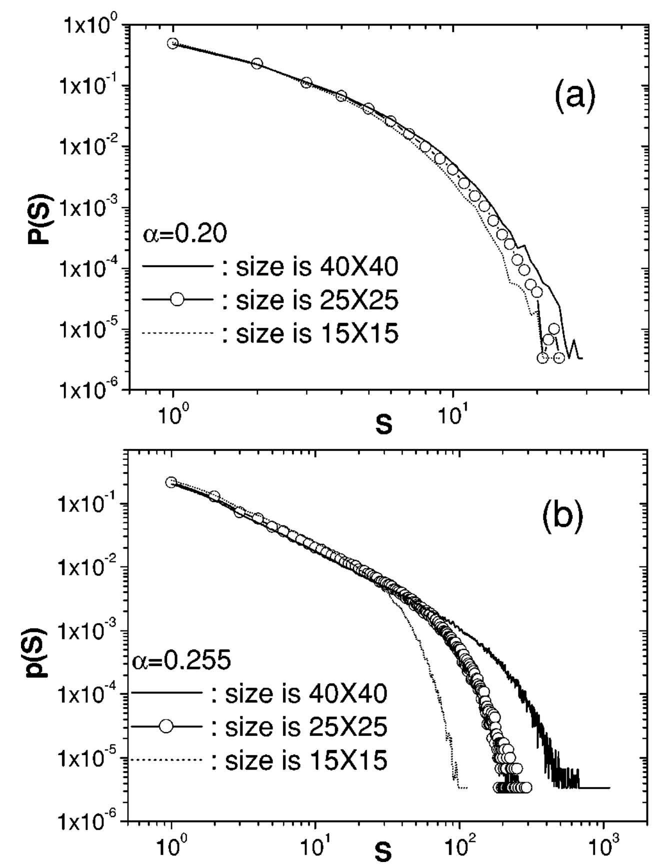

FIG. 4. (a) Simulation results for the probability of the avalanche size $P(S)$ as a function of size S with $\alpha=0.20$. The different curves refer to different system sizes, $N=15$, 25, and 40. (b) The same with (a) but $\alpha=0.255$.

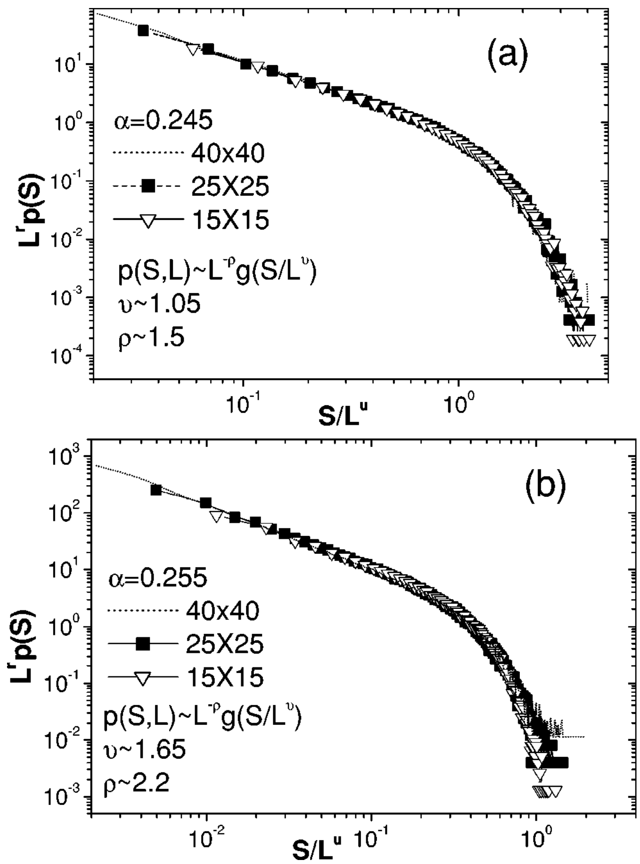

FIG. 5. (a) Data collapse of the case with the $\alpha=0.245$ and $N=15$, 25, and 40, respectively, using $P(S,L)\sim L^{-\rho}g(S/L^{v})$. (b) The same with (a) but $\alpha=0.255$.

that, with the increment of size L, the range of power-law behavior increases. We also perform a finite-size-scaling analysis of our data. We make the fit that $P(S,L)$ scales with system size L as

$$P(S,L)\sim L^{-\rho}g(S/L^{v}), \tag{4}$$

where g is the so-called universal scaling function. As can be seen in Figs. 5(a), and 5(b), the finite-size scaling works well and $\rho\sim 1.5$, $v\sim 1.05$ when $\alpha=0.245$, and $\rho\sim 2.2$, $v\sim 1.65$ when $\alpha=0.255$; it is a kind of finite-size effect and is similar to the results of the OFC model [2].

From Figs. 4 and 5, we find that the cutoff at large avalanche sizes scale with the system size L in a certain range of the parameter $(0.24<\alpha<0.26)$. This can show the criticality of the system. Some think that it can be treated as a criterion that the system reaches the SOC state [14].

D. Boundary effects

In the study of SOC, the influence of boundary conditions have been widely discussed [2,11–14]. It is believed that for the fundamental mechanism producing SOC in the OFC model that the boundaries act as inhomogeneities that frustrate the natural tendency of the model to synchronize. Indeed, it has been shown that, with period boundary condition, the avalanches of the model are localized and criticality is not observed [12–14]. So we want to investigate the boundary effects of our model.

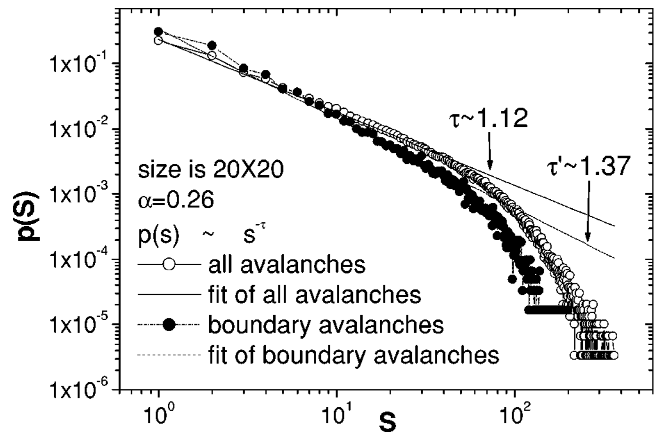

FIG. 6. Simulation results for all and boundary avalanches in the 20×20 lattice with open boundary conditions and $\alpha=0.26$.

We first perform numerical simulations in order to sample all and boundary avalanche (avalanches starting from borders) size distributions (Fig. 6); here we use open boundary conditions, respectively. We can see that all the avalanches are distributed as $P(S)\propto S^{-\tau}$, where $\tau\approx 1.12$, and the boundary avalanches are distributed as $P(S)\propto S^{-\tau'}$, $\tau'\approx 1.37$. This result is similar to that of Ref. [15] and can be treated as evidence that open boundaries create inhomogeneities.

In Fig. 7, we draw the avalanche size distributions with open and period boundary conditions, respectively. Contrary to previous models, with the period boundary condition, the avalanches are not localized but obey the power-law and the probability of large scale avalanches is larger than that with open boundary conditions. It looks like the effective size of the lattice is large with period boundary conditions.

The reason for the contrast may be the different driving rules. The global perturbation driving rule makes the OFC model a deterministic system and our local perturbation driving rule brings some randomness to our model. It may be the randomness frustrates the tendency of our model to a localized state, which it would otherwise reach with a period boundary condition.

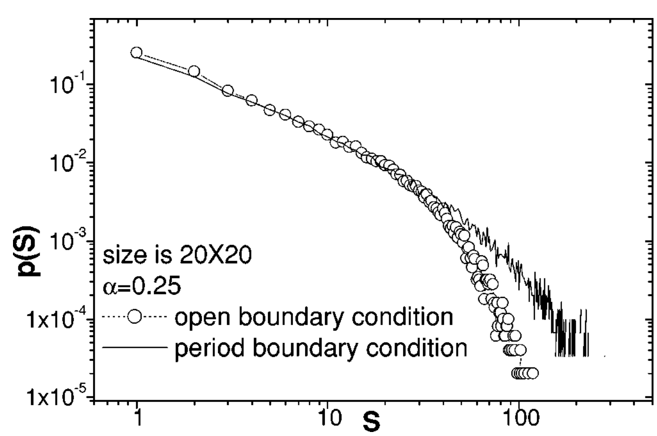

FIG. 7. $P(S)$ as a function of S for a 20×20 system with open and period boundary conditions, respectively, $\alpha=0.25$.

IV. CONCLUSION AND DISCUSSION

In conclusion, based on the standard SOM neural network model, we introduce a kind of coupled map lattice system to simulate the associative memory process of the brain. Our system is different from previous systems, because it has a kind of local perturbation driving rule and a more detailed integrate-and-fire mechanism. It can emerge SOC behavior in a certain range of system parameters. More importantly, when the influence of synaptic plasticity is adequately considered, we can find that our system's learning process plays a promotive role in the emergence of SOC behavior. We also find the average activity level of synapses have an important influence on the SOC behavior. In addition we have analyzed the influence of various factors of the model.

Our work just attempts to indicate some relations between SOC behavior and brain dynamical processes. It might provide an approach for analyzing the collective behavior of neuron populations in the brain. Based on SOC behavior, maybe we can describe the information process of the brain as follows. When an outer pattern (external signal) is input into the brain, after the learning process, the brain is trained to operate at the SOC state, the neuron populations in the brain behave with SOC behavior, the pattern is stored in the brain as a SOC attractor, and the associative memory can be designed as the process of the input pattern evolving into the attractor. But our model is only a very simple simulation of the brain and many details of neurobiology are ignored. For this reason there is still a lot of work to do.

ACKNOWLEDGMENT

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